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A classification of two-dimensional quasi-periodic tilings obtained with the grid method

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Abstract. Two-dimensional quasiperiodic tilings (QPT) obtained from n grids are classified into local isomorphism (LI) classes by using the invariants of the grids. The number of the invariants is given by $n-\phi(n)$ if n is odd or $n-\phi(2n)$ if n is even, where ϕ is Euler's function in number theory. All the QPT obtained from n grids with $n=2^k (k\geq 2)$ belong to a single LI class, whose point symmetry is D_{2n}. When n is not a power of 2, the QPT are classified into several LI classes. Of the LI classes, two have the highest point symmetry, D_{2n}; one of the two classes is associated with n grids in which all the invariants vanish and the other with n grids in which all of them take $\frac{1}{2}$. In the case of an odd n, there is also one continuous series of QPT with point symmetry D_n. We present also a general formula of the tile statistics of the quasiperiodic tilings obtained with the grid method.

1. Introduction

Two-dimensional quasiperiodic tilings (QPT) are of current interest in connection with quasi-two-dimensional quasicrystals, e.g. an octagonal one (Wang *et al* 1987), decagonal ones (Bendersky 1985, Fung *et al* 1986) and dodecagonal ones (Ishimasa *et al* 1985, Kuo 1987).

A simple method of obtaining an n-gonal (or $2n$-gonal) QPT of a plane ($n\geq 5$) is to construct it as the dual lattice of an n grid (de Bruijn 1981, Levine and Steinhardt 1986); an n grid, G_n, is the union, $G_n^{(0)}\cup G_n^{(1)}\cup\ldots\cup G_n^{(n-1)}$, of simple grids, $G_n^{(i)}$, with $G_n^{(i)}=G_n^{(i)}(\gamma_i)\equiv\{x|x\in E_2;\ \ e_i\cdot x+\gamma_i=k\in Z\}$, where E_2 is the two-dimensional Euclidean space (a two-dimensional real vector space), $e_i=(\cos i\theta,\sin i\theta)$ with $\theta=2\pi/n$ for odd n (or π/n for even n) and the γ_i are constants. The grid vectors $e_0, e_1,\ldots, e_{n-1}$ point the vertices of a regular n-gon (or a half the vertices of a regular $2n$-gon for even n). A crossing point between two grid lines belonging to $G_n^{(i)}$ and $G_n^{(j)}$ $(j>i)$ yields a rhombic tile whose sides are parallel to e_i or e_j; the four inner angles of the rhombus are given by $(j-i)\theta$ or its supplementary angle. The acute inner angles of a rhombic tile in the tiling have the form $k\pi/n$ in which k takes $1, 2,\ldots, [n/2]$; the number of different kinds of rhombi is given by $n/2$. We shall denote by T_k the rhombic tile whose acute inner angles are equal to $k\pi/n$. It is well known that the QPT obtained from an n grid are obtained, alternatively, with the projection method from a simple hypercubic lattice in n dimensions (de Bruijn 1981, Gähler and Rhyner 1986, Katz and Duneau 1986, Ishihara 1987, Ishihara *et al* 1988).

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The case $n=4$ gives an octagonal QPT known as Ammann tiling (Grunbaum and Shephard 1986) and the case $n=6$ gives two local isomorphism classes (LI classes) of dodecagonal QPT (Ishihara 1987, Ishihara *et al* 1988, Niizeki 1988). The most familiar case, $n=5$, gives Penrose tiling (de Bruijn 1981) and its generalised versions (Pavlovitch and Kléman 1987). The original Penrose tiling is obtained only when $\gamma = \gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4 \equiv 0 \bmod \boldsymbol{Z}$. This tiling has a decagonal macroscopic point symmetry represented by D_{10}, the dihedral group of order 20. On the other hand, the symmetries of the generalised Penrose tilings have not yet, to the author's knowledge, been completely established (see, however, the end of § 8).

The investigation of the cases $n=7$, 8 and 9 is only at a rudimentary stage. The purpose of the present paper is to report a complete result on classification of the QPT obtained with the grid method.

We will develop our argument in earlier sections by concentrating on the case of an odd-n grid; the case of an even-n grid will be discussed in § 6.2. In § 2 we investigate linear dependence among the grid vectors and their transformation properties under a symmetry operation. We enumerate the invariant(s) of an odd-n grid in § 3. We classify nonagrids in § 4 on the basis of symmetry considerations. We classify QPT obtained from nonagrids into LI classes in § 5. In § 6, we extend the results of §§ 4 and 5 to general cases. We present in § 7 a general formula of the tile statistics of a QPT obtained with the grid method. We discuss in § 8 related subjects, especially the role of the invariants of an n grid in the projection method which is a complementary method to the grid method.

2. Linear dependence among the grid vectors and their transformation properties

2.1. Linear dependence

We can identify E_2 with the complex plane and a vector in E_2 with the corresponding complex number. Then, grid vectors, $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$, of an odd-$n$ grid are identified with complex numbers, $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$, with $\zeta = \exp(2\pi i/n)$. These complex numbers are not, however, linearly independent over $\boldsymbol{Q}$, the field of real rational numbers, nor, accordingly, over $\boldsymbol{Z}$, the integral domain of integers. A simple relationship is $1+\zeta+\ldots+\zeta^{n-1}=0$. If n is equal to 5, 7 or any other prime number, there are no other independent relationships representing linear dependence among the ζ^i over $\boldsymbol{Z}$. Therefore, $n-1$ of the n-grid vectors are linearly independent over $\boldsymbol{Z}$ for these cases.

On the other hand, we have three independent linear relationships for the case of $n=9$, i.e. $1+\zeta^3+\zeta^6=0$, $\zeta+\zeta^4+\zeta^7=0$ and $\zeta^2+\zeta^5+\zeta^8=0$; the relationship $1+\zeta+\ldots+\zeta^8=0$ follows from the other three. In fact, the latter two among the three follow the first as algebraic relationships. The first relationship originates in the fact that the cyclic group $C_9=\{1, \zeta, \ldots, \zeta^8\}$ has $C_3=\{1, \zeta^3, \zeta^6\}$ as a subgroup. The index of C_3 in C_9 is three and the three linear relationships among the ζ^i correspond to the three cosets, C_3, ζC_3 and $\zeta^2 C_3$. It is important that the three linear relationships form a 'basis set' of linear relationships; any linear relationship among the ζ^i over $\boldsymbol{Z}$ can be represented as a linear combination of the three with integer coefficients. It follows that only six of the the nine grid vectors are linearly independent over $\boldsymbol{Z}$.

For a general n, we obtain by using number theory (for number theoretical points in this paper, see Hardy and Wright (1979)) that the number of linearly independent grid vectors over $\boldsymbol{Z}$ is given by $m=\phi(n)$ where ϕ is Euler's function. Then the number of independent linear relationships among the n-grid vectors is given by $l=n-m$. Note that m is even but l is odd.

If the grid vectors are represented by the ζ^i, we can take as the linear relationships the equations $\zeta^i P_n(\zeta)=0$, $i=0,1,\ldots,l-1$, where $P_n(x)=a_0+a_1x+\ldots a_{m-1}x^{m-1}+x^m$, with the a_i being integers ($a_0=1$), is the n-cyclotomic polynomial. These linear relationships form a basis set. It follows that $1, \zeta, \ldots, \zeta^{m-1}$ are linearly independent over z and that ζ^j with $j \geq m$ is represented by a linear combination with integer coefficients of $1, \zeta, \ldots, \zeta^{m-1}$. We should remark that the number, l, is considered, alternatively, to represent the degree of degeneracies in the set of the grid vectors.

2.2. Transformation properties among the grid vectors

Let r_n be a rotation of the plane by $2\pi/n$. Then the grid vectors are transformed cyclically as $r_n \boldsymbol{e}_i = \boldsymbol{e}_{i+1}$, $i=0,1,\ldots,n-1$, with $\boldsymbol{e}_n \equiv \boldsymbol{e}_0$. Thus r_n is represented by an n-dimensional cyclic-permutational matrix, which is a unimodular matrix. The characteristic equation of the matrix is given by x^n-1. On the other hand, the first m-grid vectors are transformed linearly among themselves by r_n. More exactly, we obtain $r_n \boldsymbol{e}_i = \boldsymbol{e}_{i+1}$, $i=0,1,\ldots,m-2$, and $r_n \boldsymbol{e}_{m-1} = -a_0\boldsymbol{e}_0 - a_1\boldsymbol{e}_1 - \ldots - a_{m-1}\boldsymbol{e}_{m-1}$, where the a_i are the coefficients of the polynomial $P_n(x)$. Thus r_n is also represented by an m-dimensional unimodular matrix. The n-cyclotomic polynomial is nothing but the characteristic polynomial of this matrix. This is an irreducible unimodular matrix because $P_n(x)$ is irreducible over $\boldsymbol{Z}$. In fact, this m-dimensional representation is an irreducible component of the above mentioned n-dimensional one; x^n-1 is factorised as $P_n(x)Q_n(x)$ where $Q_n(x)=c_0+c_1x+\ldots+c_{l-1}x^{l-1}+x^l$, with the c_i being integers, is a polynomial, which we shall call the complementary n-cyclotomic polynomial. Note that $c_0=-1$ and $c_l=1$. Note also that $c_1+c_2+\ldots+c_l=1$ because $Q_n(1)=0$. It is well known that $Q_n(x)$ is factorised into cyclotomic polynomials of lower orders;

$$Q_n(x)=\prod_{d|n}{}' P_d(x) \tag{1}$$

where the multiplication is restricted to other divisors of n than n itself. $Q_n(x)$ takes a simple form when $n=p^k$ with p being an odd prime number, i.e., $Q_n(x)=x^l-1$ with $l=p^{k-1}$.

3. The invariants of odd-n grids

The grid parameter γ_i in the simple grid $G_n^{(i)}(\gamma_i)$ specifies a relative positional relationship of the grid to the origin of E_2. If the simple grid is translated by a vector $\boldsymbol{t}$, γ_i changes as $\tilde{\gamma}_i=\gamma_i-\boldsymbol{e}_i\cdot\boldsymbol{t}$. Obviously, an n grid and its translated version give an identical QPT except for a translation. On the other hand, since the γ_i are determined modulo $\boldsymbol{Z}$, there is a one-to-one correspondence between the set of all the n grids and the n-dimensional torus, $\boldsymbol{T}^n=\boldsymbol{R}^n/\boldsymbol{Z}^n$.

Let $m_0\boldsymbol{e}_0+m_1\boldsymbol{e}_1+\ldots+m_{n-1}\boldsymbol{e}_{n-1}=0$ be a linear relationship among the grid vectors over Z. Then $\gamma=m_0\gamma_0+m_1\gamma_1+\ldots+m_{n-1}\gamma_{n-1}$ is invariant against any translation of the n grid. We can assume without loss of generality that the m_i are relative primes, i.e. they have no non-trivial common divisors. Then γ is determined modulo $\boldsymbol{Z}$ and we can assume that $-\frac{1}{2}<\gamma\leq\frac{1}{2}$. In the case where we have several invariants, the n

grid is classified by the values of the invariants $\gamma, \gamma', \ldots, \gamma^{(l-1)}$ which are associated with the basis set of the linear relationships, $\zeta^i P_n(\zeta)=0$, $i=0, 1, \ldots, l-1$. We shall denote by $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$ the family of n grids whose invariants take common values specified by $\gamma, \gamma', \ldots, \gamma^{(l-1)}$.

If the values of the l invariants are fixed, grid parameters $\gamma_m, \gamma_{m+1}, \ldots, \gamma_{n-1}$ are related to $\gamma_0, \gamma_1, \ldots, \gamma_{m-1}$ by

$$\gamma_i = \sum_{j=0}^{m-1} k_{ij}\gamma_j + \delta_i \qquad i=m, m+1, \ldots, n-1 \tag{2}$$

where the k_{ij} are integers and the δ_i are constants depending linearly on the values of the invariants. Hence, there is a one-to-one correspondence between the set of all the n grids in $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$ and the m-dimensional torus $\boldsymbol{T}^m$. Moreover, from the fact that the k_{ij} in (2) are integers, we can conclude that $\boldsymbol{T}^m$ is a closed submanifold of $\boldsymbol{T}^n$, which represents the set of all the n grids.

Two QPT belong to the same local isomorphism (LI) class or translational LI class if and only if any portion of one of the two can be completely superposed onto a portion of the other by a congruence transformation or a simple translation, respectively, and vice versa; a congruence transformation is a combined transformation of a translation, a rotation and/or a reflection.

If two QPT obtained from two n grids belong to the same translational LI class, there must be translations of the grid of one QPT which are arbitrarily close to the grid of the other one (and vice versa) so that the two grids must have the same invariants and thus belong to the same family, $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$. Conversely, all the QPT obtained from n grids in $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$ are shown to belong to a common translational LI class. A proof of this proposition is presented in appendix 1. Thus, there exists a one-to-one correspondence between the set of all the translational LI classes of QPT obtained from n grids and the l-dimensional torus, $\boldsymbol{T}^l = \boldsymbol{T}^n/\boldsymbol{T}^m$.

Since a simple grid $G_n^{(i)}(\gamma_i)$ is transformed by r_n into $G_n^{(i+1)}(\gamma_i)$, the grid parameters γ_i are transformed by r_n as $r_n\gamma_i = \gamma_{i-1}$, $i=0, 1, \ldots, n-1$, with $\gamma_{-1} \equiv \gamma_{n-1}$. The invariants are linear forms with respect to the grid vectors and are transformed by r_n too. The transformed linear forms are, obviously, also invariants. Therefore, the set of l invariants, $\gamma, \gamma', \ldots, \gamma^{(l-1)}$, form a unimodular representation of the cyclic group C_n generated by r_n; $C_n \simeq \{1, \zeta, \ldots, \zeta^{n-1}\}$.

4. A classification of nonagrids

Prior to investigating the classification problem of n grids for a general n, we investigate the case of nonagrids. In this case, we have three invariants γ, γ' and γ'' corresponding to the three cosets C_3, ζC_3 and $\zeta^2 C_3$, respectively. It follows that $r_9(\gamma, \gamma', \gamma'') = (\gamma'', \gamma, \gamma')$. Accordingly, QPT obtained from nonagrids in $G_9[\gamma, \gamma', \gamma'']$ has a macroscopic nonagonal point symmetry if $\gamma=\gamma'=\gamma''$. Otherwise, they have a trigonal point symmetry because $r_3(=r_9^3)$ leaves γ, γ' and γ'' invariant. The trigonal symmetry is a crystallographic point symmetry and we do not have much interest in this case. Therefore, we will restrict our arguments to the nonagonal QPT, i.e. the case where $\gamma=\gamma'=\gamma''$. We shall denote the corresponding family of nonagrids simply by $G_9[\gamma]$.

The reflection, s, of a vector in E_2 with respect to the real axis is equivalent to taking the complex conjugate of the corresponding complex number. Therefore, the three cosets, C_3, ζC_3, and $\zeta^2 C_3$ are transformed by s to C_3, $\zeta^2 C_3$ and ζC_3, respectively. Since the values of the three invariants are common in $G_9[\gamma]$, we obtain $sG_9[\gamma] = G_9[\gamma]$.

Thus the QPT obtained from nonagrids in $G_9[\gamma]$ have a macroscopic point symmetry represented by D_9 ($=C_9+sC_9$).

Since the grid vectors are inverted by r_2, the rotation by $\pi(=180°)$, we obtain $r_2G_9[\gamma]=G_9[-\gamma]$; that is, $G_9[\gamma]$ and $G_9[-\gamma]$ are related rotationally to each other. Therefore, we can restrict our consideration to the case where $0\leq\gamma\leq\frac{1}{2}$. Moreover, we find that $r_2G_9[0]=G_9[0]$ and $r_2G_9[\frac{1}{2}]=G_9[\frac{1}{2}]$; the latter is true because γ is determined modulo Z. Thus, we can conclude that QPT obtained from $G_9[0]$ and $G_9[\frac{1}{2}]$ have a macroscopic point symmetry represented by D_{18} ($=D_9+r_2D_9$).

Note that most of QPT with a nonagonal macroscopic point symmetry do not have any centre with an exact nonagonal symmetry; by a nonagonal macroscopic symmetry, we mean only that a QPT with this symmetry belongs to the same translational LI class as that of its rotated version by $2\pi/9$. A nonagonal QPT obtained from any nonagrid in $G_9[\gamma]$ has, however, an infinite number of vertices with a local nonagonal symmetry with a finite density.

5. Nonagonal and 18-gonal QPT

A nonagrid can be divided, naturally, into three trigrids corresponding to the three cosets, C_3, ζC_3 and ζ^2C_3. Trigrids associated with C_3 are classified into families as $G_3[\gamma]$ with $\gamma=\gamma_0+\gamma_3+\gamma_6$ ($0\leq\gamma\leq\frac{1}{2}$). We show in figure 1 trigrids with γ being equal to 0, $\frac{1}{3}$ and $\frac{1}{2}$. Trigrids in $G_3[0]$ are triangular grids and those in $G_3[\frac{1}{2}]$ are Kagomé grids. These grids have a point symmetry of D_6. On the other hand, trigrids in $G_3[\gamma]$ with $0<\gamma<\frac{1}{2}$ have a trigonal symmetry only. Note that triangular grids in $G_3[0]$ are

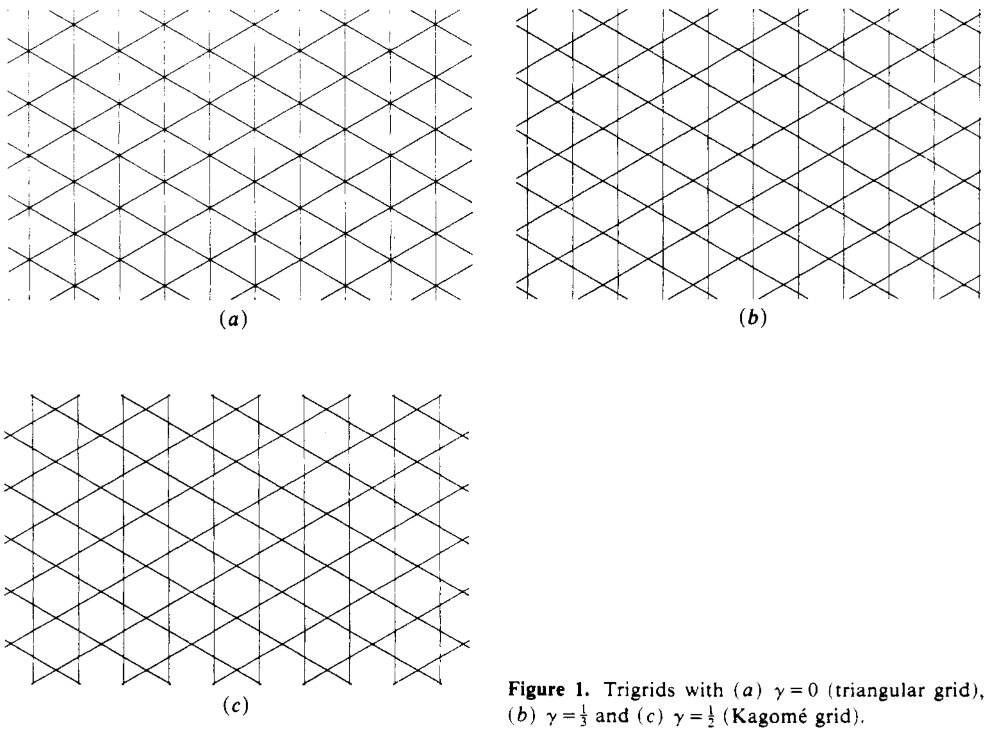


Figure 1. Trigrids with (*a*) $\gamma=0$ (triangular grid), (*b*) $\gamma=\frac{1}{3}$ and (*c*) $\gamma=\frac{1}{2}$ (Kagomé grid).

singular; every crossing point of a triangular grid is a triple crossing point of three grid lines.

A nonagrid in $G_9[\gamma]$ is a superposition of three trigrids which belong to $G_3[\gamma]$, $r_9G_3[\gamma]$ and $r_9^2G_3[\gamma]$. In particular, a nonagrid in $G_9[0]$ is a triple triangular grid and the one in $G_9[\frac{1}{2}]$ a triple Kagomé grid.

We can take a QPT with an exact nonagonal or 18-gonal point symmetry as a representative of each LI class of nonagonal QPT. A nonagrid in $G_9[\gamma]$, $0\leq\gamma\leq\frac{1}{2}$, has an exact nonagonal symmetry if all the grid parameters γ_i are equal. We can assume without loss of generality that $0\leq\gamma_i\leq\frac{1}{2}$ for all i. From this assumption and the relationship $3\gamma_0$ $(=\gamma_0+\gamma_3+\gamma_6)\equiv\gamma$ modulo Z, we obtain $\gamma_i=\gamma/3$ or $(\gamma+1)/3$ with $i=0$–8. Note, however, that the nonagrid with $\gamma_i=0$, $i=0$–8, is the most singular grid, in which nine grid lines intersect at the origin; a singular grid gives rise to a spontaneous symmetry breaking in the corresponding QPT (de Bruijn 1981). Therefore, an LI class of QPT obtained from nonagrids in $G_9[\gamma]$ includes two QPT with an exact nonagonal symmetry if $0<\gamma\leq\frac{1}{2}$ but only one if $\gamma=0$.

We show in figure 2 the nonagonal QPT obtained from a nonagrid with $\gamma_i=\frac{1}{9}$, $i=0$–8, which belongs to $G_9[\frac{1}{3}]$, and in figure 3(a, b) the 18-gonal QPT obtained from those with $\gamma_i=\frac{1}{6}$, $i=0$–8, and $\gamma_i=\frac{1}{2}$, $i=0$–8, both of which belong to $G_9[\frac{1}{2}]$. The exact point symmetry of the QPT in figure 2 or figure 3(a) is D_9 but that in figure 3(b) is D_{18}. Note, however, that the macroscopic point symmetry of the QPT in figure 3(a) is D_{18}, though we cannot observe it in the figure because only a small portion of the QPT is shown. Note also that any 18-gonal QPT obtained from a nonagrid in $G_9[\frac{1}{2}]$ has vertices with a local 18-gonal symmetry (as that in figure 3(b)) with a finite density, while that from a nonagrid in $G_9[0]$ has no such vertex. The basic tiles of these tilings are T_1, T_2, T_3 and T_4, whose acute inner angles are 20°, 40°, 60° and 80° respectively.

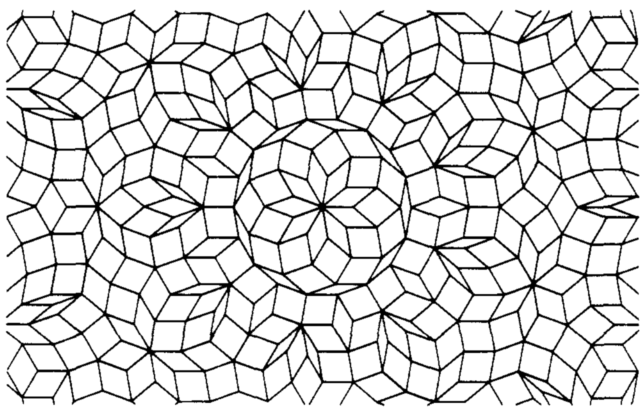
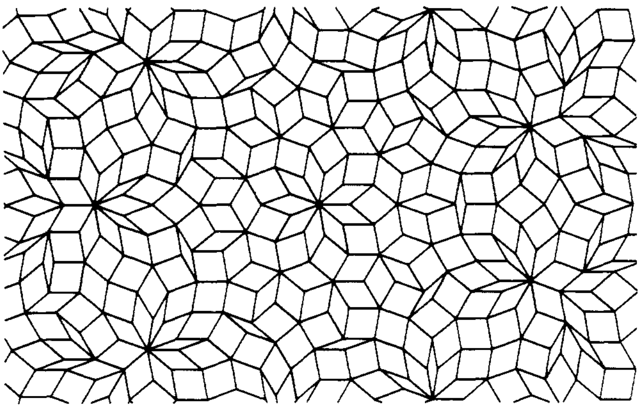
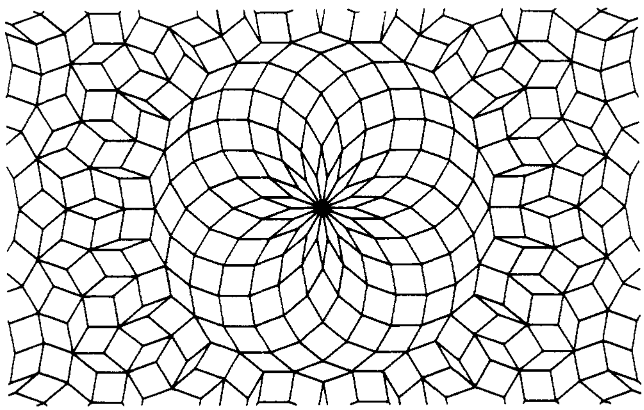


Figure 2. A nonagonal QPT obtained from a nonagrid with grid parameters $\gamma_0=\gamma_1=\ldots=\gamma_8=\frac{1}{9}$.

There is a complication in constructing a QPT from a nonagrid in $G_9[0]$ because, as well as being a trigrid in $G_3[0]$, it is also a singular grid. One way of treating this is to consider a nonagrid in $G_9[0]$ to be a limiting grid of a regular grid in $G_9[\gamma]$ in the limit $\gamma\rightarrow+0$. Then, a triple crossing point in a nonagrid in $G_9[+0]$ turns, in the dual lattice, to a triplet of three T_3, which form a regular hexagon, T_H. Unfortunately,



(a)



(b)

Figure 3. Two 18-gonal QPT obtained from triple Kagomé grids. The grid parameters (a) $\gamma_0 = \gamma_1 = \ldots = \gamma_8 = \frac{1}{6}$ and (b) $\gamma_0 = \gamma_1 = \ldots = \gamma_8 = \frac{1}{2}$.

the point symmetry of the resulting tiling is D_9 but not D_{18} because $r_2 G_9[+0]$ $(= G_9[-0]) \neq G_9[+0]$. We can change this nonagonal tiling into an 18-gonal one by substituting the hexagonal tile, T_H, for each triplet of T_3. We show in figure 4 a QPT obtained in this way from a nonagrid with $\gamma_i = \frac{1}{3}$, $i = 0$–8. The resulting tiling is composed of tiles of type T_1, T_2, T_4 and T_H. The exact point symmetry of this QPT is D_9 but its macroscopic point symmetry is D_{18}. Note that each hexagonal tile is the dual counterpart of a triple crossing point of the singular nonagrid. A hexagonal tile can take one of the three orientations corresponding to the three triangular grids forming the nonagrid.

The tiling with the hexagonal tiles is obtained, alternatively, with the projection method from a six-dimensional hyperhoneycomb lattice, which is the direct product of three identical honeycomb lattices (cf Niizeki 1988).

In summary, non-crystallographic QPT obtained from nonagrids contain two discrete LI classes with point symmetry D_{18} and one continuous series of LI classes with point symmetry D_9.

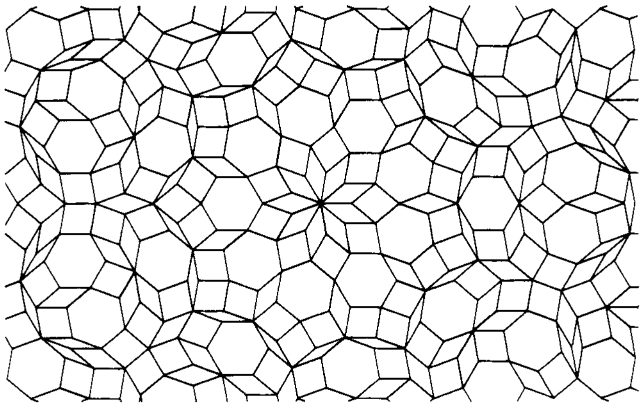


Figure 4. An 18-gonal QPT obtained from a triple triangular grid with $\gamma_0 = \gamma_1 = \ldots = \gamma_8 = \frac{1}{3}$.

6. Classifications of QPT obtained from general *n* grids

6.1. *The case of a general odd-n grid*

We begin by noting that the unimodular representation of r_n formed by the l invariants is given by the set of equations

$$r_n\gamma = \sum_{i=1}^{l} c_i\gamma^{(i-1)} \qquad r_n\gamma^{(i)} = \gamma^{(i-1)} \qquad i = 1, 2, \ldots, l-1 \tag{3}$$

where the first equality follows from the equality

$$\sum_{i=1}^{l} c_i x^i P_n(x) - P_n(x) = P_n(x)Q_n(x) \equiv 0 \bmod x^n - 1. \tag{4}$$

Thus, $Q_n(x)$ is just the characteristic polynomial of the matrix representing r_n. This matrix is a reducible unimodular matrix unless n is a prime number because $Q_n(x)$ is factorised. Since 1 is a root of $Q_n(x)$, the representation contains the identity representation, in which r_n is represented by 1. The one-dimensional space forming this identity representation is composed of l-dimensional vectors of the form $(\gamma, \gamma, \ldots, \gamma)$ with γ being any number.

It is now evident that the condition for $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$ to be invariant against r_n is given by $\gamma = \gamma' = \ldots = \gamma^{(l-1)}$. Thus, an LI class of n-gonal QPT obtained from n grids is specified by a single parameter, γ, representing the common value of the l invariants. We denote by $G_n[\gamma]$ the family of all such n grids. The QPT obtained from an n grid which does not belong to $G_n[\gamma]$ for any γ has a lower symmetry than the n-gonal point symmetry and we have no interest in it.

It is a general property of the n-cyclotonic polynomial that $P_n(x^{-1}) = x^{-m}P_n(x)$. From this we can show easily that $sG_n[\gamma] = G_n[\gamma]$ with s being the reflection with respect to the real axis. Moreover, we can show that $r_2G_2[\gamma] = G_n[-\gamma]$. Thus, we can conclude as in § 4 that two LI classes of QPT with point symmetry D_{2n} are obtained from n grids in $G_n[0]$ and $G_n[\frac{1}{2}]$ and one continuous series of LI classes of QPT with point symmetry D_n from $G_n[\gamma]$ with $0 < \gamma < \frac{1}{2}$. A QPT obtained from an n grid in $G_n[\frac{1}{2}]$ has a vertex with a $2n$-gonal local point symmetry, while any QPT obtained from an n grid in $G_n[\gamma]$ with arbitrary γ has a vertex with an n-gonal one.

The most important case is the one where $n=5$. Then, we should call the LI class of decagonal QPT obtained from pentagrids in $G_5[0]$ the Penrose class since this class is identical to that obtained by Penrose (de Bruijn 1981). The second LI class of decagonal QPT associated with the family of pentagrids, $G_5[\frac{1}{2}]$, can be called the anti-Penrose class. A QPT in this class contains a vertex with the decagonal local point symmetry but one in the Penrose class does not. We obtain also one continuous series of LI classes of pentagonal QPT from pentagrids in $G_5[\gamma]$, $0<\gamma<\frac{1}{2}$. We can obtain no other QPT with lower symmetries from pentagrids contrary to the case of nonagrids. These results are in agreement with those by Pavlovitch and Kléman (1987) although they did not mention explicitly the symmetries of the LI classes.

6.2. The case of even-n grids

In an even-n grid, the grid vectors are represented by complex numbers, 1, ζ, $\zeta^2, \ldots, \zeta^{n-1}$ with $\zeta=\exp(\pi i/n)$; note that $\zeta^n=-1$ and $\zeta^{2n}=1$. These grid vectors form a unimodular representation of r_{2n}, the rotation by π/n, and the characteristic polynomial of the representation matrix is given by x^n+1 because $r_{2n}e_{n-1}=-e_0$. The number of linearly independent grid vectors over Z is given by $m=\phi(2n)$, so that the degree of degeneracies among them is given by $l=n-m$. A basis set of linear relationships is given by $\zeta^i P_{2n}(\zeta)=0$, $i=0,1,\ldots,l-1$, with $P_{2n}(x)$ being the $2n$-cyclotomic polynomial and the corresponding invariants are denoted by γ, $\gamma', \ldots, \gamma^{(l-1)}$.

If $n=2^k (k\geqslant 2)$, we obtain $l=0$. Therefore, all the QPT obtained from 2^k grids belong to a single LI class whose point symmetry is D_{2n}. The simplest case is $n=4$ and we obtain a single LI class of octagonal QPT from tetragrids.

We shall now consider the case where n has an odd prime number as its divisor. Then, we can write $n=qn'$, where n' $(n'>1)$ is an odd integer and $q=2^k (k>0)$. Accordingly, we obtain $m=qm'$ with $m'=\phi(n')$. Using this together with the equality $x^n+1=-[(-x^q)^{n'}-1]$, we can show that $P_{2n}(x)=P_{n'}(-x^q)$ and $P_{2n}(x)Q_n(x)=x^n+1$ with $Q_n(x)\equiv -Q_{n'}(-x^q)$, where $P_{n'}$ is the n'-cyclotomic polynomial and $Q_{n'}$ the complementary one. If $n'=p^j (j\geqslant 1)$, we obtain $Q_n(x)=x^l+1$.

Using the results above and making similar arguments to those in §§ 2, 3 and 6.1, we can show that $Q_n(x)$ is the characteristic polynomial of the matrix representing r_{2n} in the unimodular representation formed by the l invariants $\gamma, \gamma', \ldots, \gamma^{(l-1)}$ and that the matrix elements are given in terms of the coefficients of $Q_n(x)$. In the special case where $n'=p^j$, we obtain $r_{2n}\gamma=-\gamma^{(l-1)}$ and $r_{2n}\gamma^{(i)}=\gamma^{(i-1)}$, $i=1,2,\ldots,l-1$. Therefore, the unimodular representation does not contain the identity representation, in which $\gamma=\gamma'=\ldots=\gamma^{(l-1)}\neq 0$. This conclusion remains true for a general n because $Q_n(1)=-Q_{n'}(-1)\neq 0$, i.e. 1 is not an eigenvalue of the unimodular matrix. Thus, $G_n[\gamma]$, the family of n grids whose invariants take a common value γ is not invariant against r_{2n}, in general. However, $G_n[0]$, which is associated with the null representation, is exceptional. There is another exceptional family, i.e. $G_n[\frac{1}{2}]$. This is true in the case, $n'=p^j$, because $-\frac{1}{2}\equiv\frac{1}{2} \bmod Z$ and the invariants are determined modulo Z. We can also prove it for a general n, as given in appendix 2.

By a similar argument to that in § 6.1 we can show that $sG_n[0]=G_n[0]$ and $sG_n[\frac{1}{2}]=G_n[\frac{1}{2}]$ with s being the reflection. Thus, these two families yield two LI classes of QPT with point symmetry D_{2n}; one is the 'Penrose class' associated with $G_n[0]$ and the other the 'anti-Penrose class' associated with $G_n[\frac{1}{2}]$. A QPT belonging to the former does not contain a vertex with a $2n$-gonal (nor even n-gonal) local point symmetry but one belonging to the latter does. In fact the QPT obtained from an n grid with

$\gamma_0 = \gamma_1 = \ldots = \gamma_{n-1} = \frac{1}{2}$ has D_{2n} as its exact point symmetry; this n grid belongs to $G_n[\frac{1}{2}]$ by the general property of a cyclotomic polynomial that $P_{2n}(1) \equiv 1 \bmod 2$.

It can be shown that only one LI class of n-gonal QPT is obtained from even-n grids, whereas one continuous series is obtained from odd-n grids. This LI class is, however, not important because we can obtain other LI classes of n-gonal QPT with much simpler structures from $\tilde{n}$ grids with $\tilde{n} = n/2$ being an integer.

In the simplest case, i.e. $n = 6$, we obtain two LI classes of dodecagonal QPT from double triangular grids and double Kagomé grids. The double triangular grids are singular and we obtain dodecagonal tilings containing a hexagonal tile in addition to a square tile and a rhombic tile whose acute inner angles are equal to 30° (Niizeki 1988). Incidentally, we remark that an LI class of hexagonal QPT is obtained from a family of hexagrids, each of which is a superposition of a triangular grid and a Kagomé grid.

7. Statistics of different kinds of tiles in a QPT obtained from an n grid

We consider the case of an odd-n grid first. The crossing points between two different simple grids in an n grid form a rhombic lattice in two dimensions. The density of the lattice points is inversely proportional to the area of the unit cell and, consequently, the density of the crossing points between $G_n^{(i)}$ and $G_n^{(j)}$ $(j > i)$ is proportional to $|\sin[(j-i)\theta]|$ with $\theta = 2\pi/n$. Since there are n pairs of simple grids which yield the same kind of tiles, the probability, P_k, of the appearance of T_k in the relevant QPT is proportional to $\sin(k\pi/n)$, i.e.

$$P_1 : P_2 : P_3 : \ldots = 1 : \sigma : (\sigma^2 - 1) : \sigma(\sigma^2 - 2) : \ldots \tag{5}$$

with $\sigma = 2\cos(\pi/n)$. For example, we obtain for $n = 5$ the known result, $P_1 : P_2 = 1 : (1 + \sqrt{5})/2$. It is remarkable that the tile statistics of the QPT obtained from an n grid are independent of a particular LI class to which it belongs.

Equation (5) applies principally to the case of an even-n grid, too, but note that in this case a square tile appears and its statistical weight in (5) has to be halved because there are only $n/2$ pairs of simple grids yielding square tiles. Thus we obtain $P_1 : P_2 = 1 : 1/\sqrt{2}$ for the case of octagonal QPT and $P_1 : P_2 : P_3 = 1 : \sqrt{3} : 1$ for the dodecagonal case.

The tile statistics of a QPT with hexagonal tiles can be obtained easily from those of the corresponding rhombic tiling because the weight of T_H is one-third of that of the rhombic tile whose acute inner angles are 60°.

The number of the vertices in a tiling with rhombic tiles (more generally, quadrilateral tiles) only is equal to the total number of the tiles; the four inner angles of a rhombus total 2π and the inner angles of rhombi joining at a vertex also total 2π.

Most of the results in this subsection can be extended to QPT obtained from a generalised grid, e.g., the double honeycomb grid (Stampfli 1986) and also to the icosahedral quasiperiodic 'tiling' in three dimensions (see, e.g., Elser 1986).

8. Discussion

The vertices of a QPT form a quasiperiodic lattice. A QPT and its associated quasiperiodic lattice have usually a common point symmetry. For example, we obtain an 18-gonal quasiperiodic lattice from a nonagrid in $G_9[0]$ or $G_9[\frac{1}{2}]$). However, there are exceptional cases; the point symmetry of a QPT obtained from a nonagrid in $G_9[+0]$ is D_9 as noted in § 4 but that of the associated quasiperiodic lattice is D_{18}. This is because the difference between $G_9[+0]$ and $G_9[-0]$ is only in the two ways of dividing a hexagonal tile, T_H, into a triplet of T_3.

A quasiperiodic lattice associated with a QPT obtained from an n grid is also obtained with the projection method (see, e.g., Gähler and Rhyner 1986) from an n-dimensional simple hypercubic lattice, L_n, which is embedded in E_n, the n-dimensional Euclidean space. The tiling space, E_2, which is a subspace of E_n, is spanned by the real and the imaginary parts of the complex vector $(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$ (Ishihara 1987). The internal space, E_{n-2}, is the orthogonal complement of E_2 in E_n. A quasiperiodic lattice is obtained by projecting the lattice points in a subset of L_n onto the tiling space; the projections of the lattice points in the subset onto E_{n-2} are required to go to a window which is a finite domain in E_{n-2}. The quasiperiodic lattice associated with a QPT obtained from an n grid is obtained also with the projection method when the window coincides with the projection of the unit cell of L_n onto E_{n-2}. We can choose arbitrarily the location of the centre of the window in the internal space and this freedom is specified by the position vector of the centre of the window. The vector, which is called a phase vector, is closely related to parameters γ_i in the corresponding n grid.

If the set of the grid vectors has no degeneracies or, equivalently, if $n = 2^k (k \geqslant 2)$, the projections of the lattice points of L_n onto E_{n-2} distribute uniformly in the window. Otherwise, they distribute only in a number of k-dimensional cross sections of the window, where $k = n - 2 - l$ $(= m - 2)$ with l being the degeneracy among the grid vectors. Which cross sections of the window are concerned depends on the phase vector and, consequently, on the values of γ_i. More exactly, the relevant cross sections are determined by the values of the l invariants. The symmetry of the quasiperiodic lattice depends on the symmetries of the cross sections, so that the invariants must take special values to obtain a quasiperiodic lattice with a high symmetry. This result is well known in the case of $n = 5$ (Henley 1986, Jarić 1986, Pavlovitch and Kléman 1987). At all events, this presents another explanation of the reason why the invariants are of fundamental importance in the classification of the QPT obtained from n grids.

In conclusion, we have succeeded in a complete classification of QPT obtained from n grids.

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Note added. A recent preprint indicates that Ishihara and Yamamoto (1988) have reached the same conclusion as the author about the symmetries of the LI classes of the QPT obtained from pentagrids.

Appendix 1

The proposition in the text is true if the set obtained from an n grid in $G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}]$ by translating it in all possible ways is dense in

$G_n[\gamma, \gamma', \ldots, \gamma^{(l-1)}] \simeq \boldsymbol{T}^m$, the m-dimensional torus. This statement can be proved by using the following lemma (theorem 5 in ch 7 of Koksma (1936)).

Lemma. Let θ_0, $\theta_1, \ldots, \theta_{m-1}$ be real numbers and assume that they are linearly independent over $\boldsymbol{Z}$. Then the image of the following mapping from $\boldsymbol{R}$ into $\boldsymbol{T}^m$ is a dense set in $\boldsymbol{T}^m$; $t \in \boldsymbol{R} \to (\theta_0 t, \theta_1 t, \ldots, \theta_{m-1} t) \bmod \boldsymbol{Z}^m$.

Our proposition is true from this lemma if we can choose a $\boldsymbol{t} = (t_1, t_2) \in E_2$ in such a way that $\theta_i = \boldsymbol{e}_i \cdot \boldsymbol{t}$, $i = 0, 1, \ldots, m-1$, satisfy the condition of the lemma. Now, let $\boldsymbol{V}_1$ (or $\boldsymbol{V}_2$) be an m-dimensional vector obtained from the first (or the second) components of the vectors, $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}$, and let $\boldsymbol{n} \in \boldsymbol{Z}^m - \{0\}$, i.e. an integer vector. Then $\boldsymbol{n} \cdot \boldsymbol{V}_1$ and $\boldsymbol{n} \cdot \boldsymbol{V}_2$ do not simultaneously vanish by the assumption that $\boldsymbol{e}_0$, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{m-1}$ are linearly independent over $\boldsymbol{Z}$, where the dot stands for the Euclidean inner product. Accordingly, $(\boldsymbol{n} \cdot \boldsymbol{V}_1, \boldsymbol{n} \cdot \boldsymbol{V}_2)$ represents a homogeneous coordinate of a point in a one-dimensional projective space $\boldsymbol{P}^1$. Obviously, $\boldsymbol{X} = \{(\boldsymbol{n} \cdot \boldsymbol{V}_1, \boldsymbol{n} \cdot \boldsymbol{V}_2) | \boldsymbol{n} \in \boldsymbol{Z}^m - \{0\}\}$ is a countable set of points in $\boldsymbol{P}^1$. Let $y = (y_1, y_2)$ by any point in $\boldsymbol{P}^1 - \boldsymbol{X} (\neq \phi)$. Then, $\boldsymbol{t} = (-y_2, y_1)$ satisfies the requirement mentioned above.

Appendix 2

We begin by noting that $Q_n(-1) \equiv -Q_{n'}(1)\ (=0) \bmod 2$, so that $Q_n(1) \equiv 0 \bmod 2$. Therefore, if we consider everything in modulo 2, the l-dimensional vector $(1, 1, \ldots, 1)$ is an eigenvector of the matrix representing r_{2n} in the l-dimensional unimodular representation. Then, the l-dimensional vector $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is also its eigenvector if we consider it in modulo Z. This completes the proof.

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